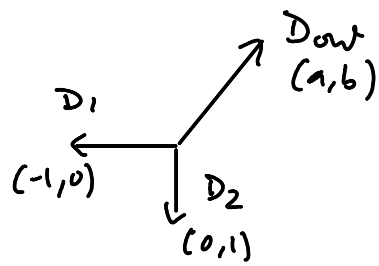
M. Gross – 28/5/08 – The tropical vertex (continued)

Last time, saw that instanton corrections are encoded by <u>scattering diagrams</u>

<u>Q</u>: <u>enumerative</u> interpretation of the <u>coefficients</u> along the rays?

Consider X a weighted proj. space given by a fan



Pick $d_1, d_2 > 0$ and $S_1 \subseteq D_1$, $S_2 \subseteq D_2$
$\#S_1 = d_1$ $\#S_2 = d_2$

<u>Def</u>: Let $N_d = \#$ maps $\varphi: \mathbb{P}^1 \to X$ up to reparametrization

s.t. 1) whenever $\varphi(p) \in D_i$, $i = 1, 2$, then $\varphi(p) \in S_i$ and φ is transversal to D_i at $\varphi(p)$

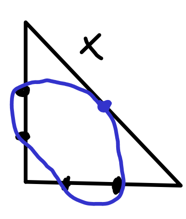
2) $\exists!\, q \in \mathbb{P}^1$ s.t. $\varphi(q) \in D_{out}$

3) the intersection multiplicity of φ with D_{out} at q is d.

<u>Rmk</u>: φ can hit D_i in any number of pts – can hit the same point of S_i several times, or not at all

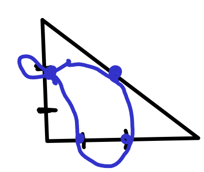
$\Rightarrow$ this is a sum of GW invariants over certain homology classes in $\hat{X}$ = blowup of X along $S_1 \cup S_2$ (also includes multiple covers)

e.g.:



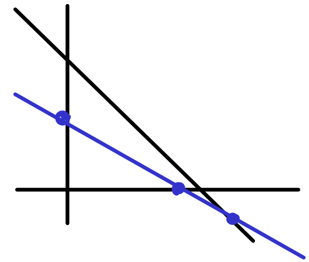
$\longleftrightarrow$ $2H - \Sigma E_i$ in $\hat{X}$

but also allow



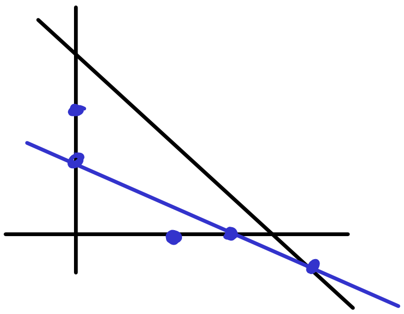
$2H - 2E_1 - E_3 - E_4$

Ex: $d_1 = d_2 = 1$, $(a,b) = (1,1)$



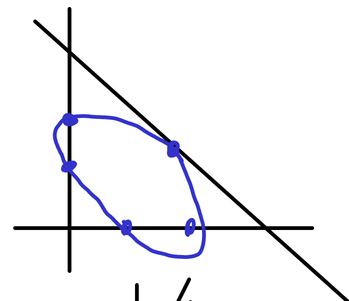
$N_1 = 1$

$d_1 = d_2 = 2$, $(a,b) = (1,1)$



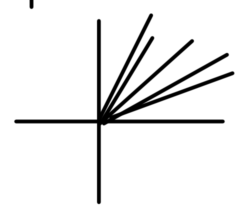
$N_1 = 4$ 4 lines: e.g.

$N_2 = 2$ 2 conics tangent to hypothenuse



$N_d = 0$ for $d \geq 3$

Compare with coeffts of $(1,1)$-ray in



$d_1 = d_2 = 3$, $(a,b) = (1,1)$

$N_1 = 9$ (lines) $\leftarrow$ choose 1 pt in S_1, 1 pt in S_2

$N_2 = 3 \cdot 3 \cdot 2 = 18$ (conics) $\leftarrow$ choose 2 pts in S_1, 2 pts in S_2

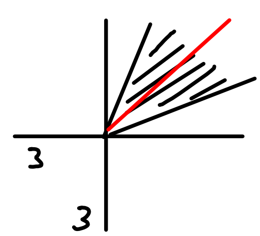
then $\exists\, 2$ conics tgt to D_{out}

$N_3 = 18 + 36 = 54$ nodal cubics

hit all 6 pts of S_i — have a node at one pt of S_i

$N_4 = 252$, ...

Compare with coeffts of $(1,1)$ ray in



<u>Theorem:</u> Let $\mathfrak{D}$ = scattering diagram with two lines w/ attached functions $(1+tx^{-1})^{d_1}$ and $(1+ty^{-1})^{d_2}$ and $\mathfrak{D}'$ = output of Kontsevich-Soibelman lemma.

Then the function f_{out} attached to ray along primitive vector (a,b) is

$$\log f_{out} = \sum_{d=1}^{\infty} d N_d \, t^{d(a+b)} x^{-da} y^{-db}$$

In addition, contribution of mult. covers of a single curve tangent to D_{out} at order d to $\log f_{out}$ is:

mult. cover contributions: if $\exists!$ simple curve then N_{kd}'s are combinatorial coefficients in these formulas.

$$\sum_{k=1}^{\infty} k d \binom{(d-1)k-1}{k-1} \frac{t^{dk(a+b)} x^{-kda} y^{-kdb}}{k^2}$$

for $d=1$, say $\binom{-1}{k-1} = (-1)^{k-1}$ and get

$$\sum_{k=1}^{\infty} k (-1)^{k-1} \frac{t^{k(a+b)} x^{-ka} y^{-kb}}{k^2}$$

$\rightsquigarrow$ Can rewrite $f_{out} = \prod_{d=1}^{\infty} G_d(t^{a+b} x^{-a} y^{-b})^{I_d}$

where $G_d(q) = \left(\sum_{k=0}^{\infty} \frac{1}{(d-2)k+1} \binom{(d-1)k}{k} q^{kd} \right)^d$

here $G_1(q) = 1+q$, $G_2(q) = \frac{1}{(1-q^2)^2}$

and I_d = instanton count (# simple curves)